

Light-Light Scattering

Naohiro Kanda^{1,*}

¹*Department of Physics, Faculty of Science and Technology, Nihon University, Tokyo, Japan*
(Dated: January 27, 2013)

For a long time, it is believed that the light by light scattering is described properly by the Lagrangian density obtained by Heisenberg and Euler. Here, we present a new calculation which is based on the modern field theory technique. It is found that the light-light scattering is completely different from the old expression. The reason is basically due to the unphysical condition (gauge condition) which was employed by the QED calculation of Karplus and Neumann. The correct cross section of light-light scattering at low energy of ($\frac{\omega}{m} \ll 1$) can be written as $\frac{d\sigma}{d\Omega} = \frac{1}{(6\pi)^2} \frac{\alpha^4}{(2\omega)^2} (3 + 2\cos^2\theta + \cos^4\theta)$.

PACS numbers: 11.10.Gh, 12.20.Ds, 13.40.Em

I. INTRODUCTION

It has been believed that photon-photon scattering can be described by the Lagrangian density of Heisenberg and Euler [1][2][3]. In addition, this cross section is confirmed by the QED calculation of Karplus and Neumann [4][5][6]. However, the calculation by Karplus and Neumann is not reliable since they have put some additional conditions on the QED calculation so as to reproduce the result obtained by Heisenberg and Euler.

In this respect, it is important that the proper treatment of the photon-photon scattering should be made again at the present stage. The calculation of photon-photon scattering itself is straightforward since the Feynman diagrams of the fourth order perturbation calculation can be done without any difficulties. The interaction Lagrangian density of $H' = -ej_\mu A^\mu$ is gauge invariant as long as the fermion current is conserved, which is indeed the case. Therefore, the QED calculation itself has no conceptual difficulty and therefore we should carry out the S-matrix evaluation of photon-photon scattering.

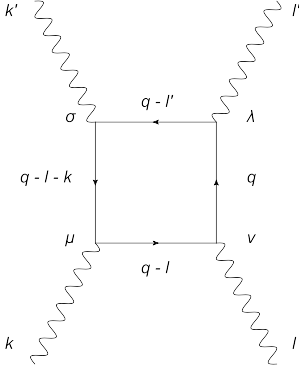


FIG. 1: M_a

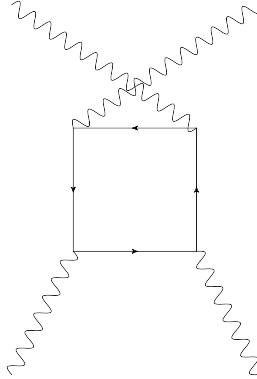


FIG. 2: M_b

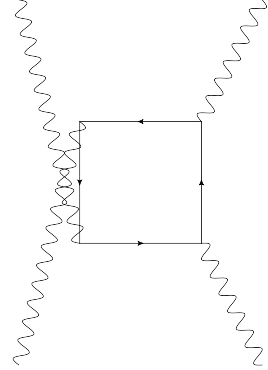


FIG. 3: M_c

*Electronic address: nkanda@phys.cst.nihon-u.ac.jp

II. FEYNMAN AMPLITUDE OF PHOTON-PHOTON SCATTERING

The Feynman amplitude of the photon-photon scattering can be written as

$$\begin{aligned} M^{rr'ss'} &= 2(M_a + M_b + M_c) \\ &= 2(\bar{M}_a^{\mu\nu\lambda\sigma} + \bar{M}_b^{\mu\nu\lambda\sigma} + \bar{M}_c^{\mu\nu\lambda\sigma})\epsilon_\mu^r(k)\epsilon_\nu^s(l)\epsilon_\lambda^{s'}(l')\epsilon_\sigma^{r'}(k') \end{aligned} \quad (1.1)$$

where we note that the Feynman amplitude is not affected by the direction of the loop momentum [7]. The amplitude M_a can be explicitly written as

$$\begin{aligned} M_a &= -\frac{(ie)^4}{(2\pi)^4} \int d^4q \frac{\text{Tr}[\gamma^\mu(\not{q} - \not{l} - \not{k} + m)\gamma^\sigma(\not{q} - \not{l}' + m)\gamma^\lambda(\not{q} + m)\gamma^\nu(\not{q} - \not{l} + m)]}{[(q-l-k)^2 - m^2 + i\varepsilon][(q-l')^2 - m^2 + i\varepsilon][q^2 - m^2 + i\varepsilon][(q-l)^2 - m^2 + i\varepsilon]} \\ &\quad \times \epsilon_\mu^r(k)\epsilon_\sigma^{r'}(k')\epsilon_\lambda^{s'}(l')\epsilon_\nu^s(l). \end{aligned} \quad (1.2)$$

Here, it should be important to note that the total Feynman amplitude of the photon-photon scattering does not have any logarithmic divergence even though the amplitude M_a alone has the logarithmic divergence. This is quite important in that the physical processes should not have any divergences, and indeed the photon-photon scattering is just the case. This strongly suggests that the evaluation of the Feynman amplitude of the photon-photon scattering process must be directly connected to the real physical process which should be observed by experiments.

III. DIVERGENCE

The amplitude M_a has the logarithmic divergence, and this can be seen when we check the large q behavior [4]. In this case, we find from (1.2)

$$M_a \sim -\frac{(ie)^4}{(2\pi)^4} \int d^4q \frac{4}{3} \frac{1}{(q^2)^2} \{g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} - 2g^{\nu\sigma}g^{\mu\lambda}\} \epsilon_\mu^r \epsilon_\nu^r \epsilon_\lambda^{s'} \epsilon_\sigma^{r'} \quad (2.1)$$

which has obviously the logarithmic divergence. However, if we add all of the amplitude together, then we find from (1.1)

$$\begin{aligned} M &\sim -2\frac{(ie)^4}{(2\pi)^4} \frac{4}{3} \int d^4q \frac{1}{(q^2)^2} \{g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} - 2g^{\nu\sigma}g^{\mu\lambda} \\ &\quad + g^{\mu\nu}g^{\sigma\lambda} + g^{\mu\lambda}g^{\nu\sigma} - 2g^{\nu\lambda}g^{\mu\sigma} \\ &\quad + g^{\sigma\nu}g^{\lambda\mu} + g^{\sigma\mu}g^{\nu\lambda} - 2g^{\nu\mu}g^{\sigma\lambda}\} \epsilon_\mu^r \epsilon_\nu^r \epsilon_\lambda^{s'} \epsilon_\sigma^{r'} \\ &= 0. \end{aligned} \quad (2.2)$$

This means that the total amplitude has no divergence at all because of the cancellation, and it is indeed finite. Therefore, we do not have to employ any specific regularization scheme, and thus the evaluation is very reliable.

IV. DEFINITION OF POLARIZATION VECTOR

Here, we take the polarization vector as defined by Lifshitz [4]

$$\begin{aligned} \epsilon_1^{(1)} &= \epsilon_2^{(1)} = \epsilon_3^{(1)} = \epsilon_4^{(1)} = \frac{\mathbf{k} \times \mathbf{k}'}{|\mathbf{k} \times \mathbf{k}'|} & \epsilon_1^{(2)} &= \frac{1}{\omega} (\mathbf{k} \times \epsilon_1^{(1)}) = -\epsilon_2^{(2)} \\ \epsilon_4^{(2)} &= \frac{1}{\omega} (\mathbf{k}' \times \epsilon_4^{(1)}) = -\epsilon_3^{(2)} \end{aligned}$$

$\mathbf{e}_n^{(i)}$ denotes the polarization vector of photons. Here, we take the Coulomb gauge fixing $\nabla \cdot \mathbf{A} = 0$.
Each photon has the following momenta

initial state

$$\text{photon 1 : } k^\mu = (\omega, \mathbf{k}) \quad \text{photon 2 : } l^\mu = (\omega, \mathbf{l})$$

final state

$$\text{photon 3 : } l'^\mu = (\omega, \mathbf{l}') \quad \text{photon 4 : } k'^\mu = (\omega, \mathbf{k}').$$

$$\omega = |\mathbf{k}| = |\mathbf{l}| = |\mathbf{k}'| = |\mathbf{l}'|$$

Also, by noting that there is no rest system, we find

$$\mathbf{l} = -\mathbf{k} \quad \mathbf{l}' = -\mathbf{k}'.$$

V. CALCULATION OF M_a AT LOW ENERGY

Now, we carry out the calculation of M_a as an example

$$M_a = \bar{M}_a^{\mu\nu\lambda\sigma} \epsilon_\mu^r(k) \epsilon_\nu^s(l) \epsilon_\lambda^{s'}(l') \epsilon_\sigma^{r'}(k') \quad (4.1)$$

where $\bar{M}_a^{\mu\nu\lambda\sigma}$ can be written as

$$\bar{M}_a^{\mu\nu\lambda\sigma} = -\frac{(ie)^4}{(2\pi)^4} \int d^4q \frac{\text{Tr}[\gamma^\mu(\not{q} - \not{l} - \not{k} + m)\gamma^\sigma(\not{q} - \not{l}' + m)\gamma^\lambda(\not{q} + m)\gamma^\nu(\not{q} - \not{l} + m)]}{[(q-l-k)^2 - m^2 + i\varepsilon][(q-l')^2 - m^2 + i\varepsilon][q^2 - m^2 + i\varepsilon][(q-l)^2 - m^2 + i\varepsilon]}. \quad (4.2)$$

By making use of the Feynman parameter [8], we find

$$\bar{M}_a^{\mu\nu\lambda\sigma} = -\frac{(ie)^4}{(2\pi)^4} \int d^4q \, 3! \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \frac{\text{Tr}[\gamma^\mu(\not{q} - \not{l} - \not{k} + m)\gamma^\sigma(\not{q} - \not{l}' + m)\gamma^\lambda(\not{q} + m)\gamma^\nu(\not{q} - \not{l} + m)]}{[q^2 - 2q \cdot A + B + i\varepsilon]^4}$$

where

$$A^\mu \equiv l^\mu z_1 + (l' - l)^\mu z_2 + k'^\mu z_3$$

$$B \equiv 2l \cdot k z_3 - m^2$$

By putting $q - A = t$, we find

$$\begin{aligned} \bar{M}_a^{\mu\nu\lambda\sigma} = & -\frac{(ie)^4}{(2\pi)^4} 3! \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \\ & \times \int d^4t \frac{\text{Tr}[\gamma^\mu(\not{t} + \not{A} - \not{l} - \not{k} + m)\gamma^\sigma(\not{t} + \not{A} - \not{l}' + m)\gamma^\lambda(\not{t} + \not{A} + m)\gamma^\nu(\not{t} + \not{A} - \not{l} + m)]}{[t^2 - A^2 + B + i\varepsilon]^4}. \end{aligned}$$

Here, for simplicity, we define

$$\not{a} = \not{A} - \not{l} - \not{k}$$

$$\not{b} = \not{A} - \not{l}'$$

$$\not{c} = \not{A}$$

$$\not{d} = \not{A} - \not{l}$$

and thus

$$\bar{M}_a^{\mu\nu\lambda\sigma} = -\frac{(ie)^4}{(2\pi)^4} 3! \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \int d^4t \frac{\text{Tr}[\gamma^\mu(\not{p} + \not{d} + m)\gamma^\sigma(\not{p} + \not{b} + m)\gamma^\lambda(\not{p} + \not{c} + m)\gamma^\nu(\not{p} + \not{d} + m)]}{[t^2 - A^2 + B + i\varepsilon]^4}$$

In the numerator, we find $\text{Tr}[\text{odd-numbers of } \gamma\text{-matrices}] = 0$ and by noting $\int d^4t \underbrace{t^\mu t^\nu \dots t^\lambda}_{\text{odd}} f(t^2) = 0$ we find

$$\begin{aligned} \text{Tr}[\gamma^\mu(\not{p} + \not{d} + m)\gamma^\sigma(\not{p} + \not{b} + m)\gamma^\lambda(\not{p} + \not{c} + m)\gamma^\nu(\not{p} + \not{d} + m)] \\ = t^2 F^{\mu\nu\lambda\sigma} + G^{\mu\nu\lambda\sigma} + \text{Tr}[\gamma^\mu \not{p} \gamma^\sigma \not{p} \gamma^\lambda \not{p} \gamma^\nu \not{p}] \end{aligned}$$

where

$$\begin{aligned} F^{\mu\nu\lambda\sigma} &\equiv -4m^2 \{g^{\mu\sigma} g^{\lambda\nu} + g^{\mu\nu} g^{\sigma\lambda} - 2g^{\mu\lambda} g^{\sigma\nu}\} \\ &\quad - \frac{1}{2} \text{Tr}[\{\gamma^\mu \gamma^\sigma \gamma^\lambda \not{p} \gamma^\nu \not{p} + \gamma^\mu \gamma^\lambda \not{p} \gamma^\sigma \gamma^\nu \not{p} + \gamma^\mu \gamma^\sigma \not{p} \gamma^\lambda \not{p} \gamma^\nu \\ &\quad + (\gamma^\mu \not{p} \gamma^\sigma)(\gamma^\lambda \gamma^\nu \not{p} + \gamma^\nu \not{p} \gamma^\lambda + \not{p} \gamma^\lambda \gamma^\nu)\}] \\ G^{\mu\nu\lambda\sigma} &\equiv \text{Tr}[\gamma^\mu(\not{p} + m)\gamma^\sigma(\not{p} + m)\gamma^\lambda(\not{p} + m)\gamma^\nu(\not{p} + m)]. \end{aligned}$$

Now, we find

$$\bar{M}_a^{\mu\nu\lambda\sigma} = -\frac{(ie)^4}{(2\pi)^4} 3! \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \int d^4t \frac{t^2 F^{\mu\nu\lambda\sigma} + G^{\mu\nu\lambda\sigma} + \text{Tr}[\gamma^\mu \not{p} \gamma^\sigma \not{p} \gamma^\lambda \not{p} \gamma^\nu \not{p}]}{(t^2 - A^2 + B + i\varepsilon)^4}$$

and we carry out the integration and obtain [8]

$$\begin{aligned} \bar{M}_a^{\mu\nu\lambda\sigma} &= -\frac{(ie)^4}{(2\pi)^4} \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 i\pi^2 \left\{ \frac{2F^{\mu\nu\lambda\sigma}}{B - A^2} + \frac{G^{\mu\nu\lambda\sigma}}{(B - A^2)^2} \right. \\ &\quad \left. + 8(g^{\mu\sigma} g^{\lambda\nu} + g^{\mu\nu} g^{\lambda\sigma} - 2g^{\sigma\nu} g^{\mu\lambda}) \left(\ln \left| \frac{\Lambda^2}{B - A^2} \right| - \frac{11}{6} \right) \right\} \end{aligned}$$

where Λ denotes the cutoff momentum.

Here, we consider the case of $\frac{\omega}{m} \ll 1$ and expand the integration in terms of $\frac{\omega}{m}$ powers. This approximation is quite reasonable as we show it later [9].

Now, we consider the expansion in terms of $\frac{\omega}{m}$. Noting that the m never appears in k , l , k' , l' in $F^{\mu\nu\lambda\sigma}$, $G^{\mu\nu\lambda\sigma}$ but only ω appears, we can rewrite $F^{\mu\nu\lambda\sigma}$, $G^{\mu\nu\lambda\sigma}$ and obtain

$$\begin{aligned} F^{\mu\nu\lambda\sigma} &= -4m^2 Q^{\mu\nu\lambda\sigma} - \frac{1}{2} \omega^2 R^{\mu\nu\lambda\sigma} \\ G^{\mu\nu\lambda\sigma} &= \omega^4 S^{\mu\nu\lambda\sigma} + m^2 \omega^2 R^{\mu\nu\lambda\sigma} + m^4 T^{\mu\nu\lambda\sigma} \end{aligned}$$

where

$$\begin{aligned} Q^{\mu\nu\lambda\sigma} &\equiv g^{\mu\sigma} g^{\lambda\nu} + g^{\mu\nu} g^{\sigma\lambda} - 2g^{\mu\lambda} g^{\sigma\nu} \\ R^{\mu\nu\lambda\sigma} &\equiv \frac{1}{\omega^2} \text{Tr}[\{\gamma^\mu \gamma^\sigma \gamma^\lambda \not{p} \gamma^\nu \not{p} + \gamma^\mu \gamma^\lambda \not{p} \gamma^\sigma \gamma^\nu \not{p} + \gamma^\mu \gamma^\sigma \not{p} \gamma^\lambda \not{p} \gamma^\nu \\ &\quad + (\gamma^\mu \not{p} \gamma^\sigma)(\gamma^\lambda \gamma^\nu \not{p} + \gamma^\nu \not{p} \gamma^\lambda + \not{p} \gamma^\lambda \gamma^\nu)\}] \\ S^{\mu\nu\lambda\sigma} &\equiv \frac{1}{\omega^4} \text{Tr}[\gamma^\mu \not{p} \gamma^\sigma \not{p} \gamma^\lambda \not{p} \gamma^\nu \not{p}] \\ T^{\mu\nu\lambda\sigma} &\equiv \text{Tr}[\gamma^\mu \gamma^\sigma \gamma^\lambda \gamma^\nu] \end{aligned}$$

Here, $Q^{\mu\nu\lambda\sigma}$, $R^{\mu\nu\lambda\sigma}$, $S^{\mu\nu\lambda\sigma}$, $T^{\mu\nu\lambda\sigma}$ are all dimensionless. In this case, we can easily expand $Q^{\mu\nu\lambda\sigma}$, $R^{\mu\nu\lambda\sigma}$, $S^{\mu\nu\lambda\sigma}$, $T^{\mu\nu\lambda\sigma}$.

Now, we can write

$$B - A^2 = -m^2 \left\{ 1 - \frac{\omega^2}{m^2} U_a \right\}$$

$$U_a = 4 \left[z_2(z_2 - z_1 - z_3) \sin^2 \frac{\theta}{2} - z_3 z_1 \cos^2 \frac{\theta}{2} + z_3 \right]$$

and therefore we have

$$\bar{M}_a^{\mu\nu\lambda\sigma} = -\frac{(ie)^4}{(2\pi)^4} i\pi^2 \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \left\{ \frac{2F^{\mu\nu\lambda\sigma}}{B - A^2} + \frac{G^{\mu\nu\lambda\sigma}}{(B - A^2)^2} + 8Q^{\mu\nu\lambda\sigma} \left(\ln \left| \frac{\Lambda^2}{m^2} \right| - \ln |1 - \xi^2 U_a| - \frac{11}{6} \right) \right\}$$

$$\xi \equiv \frac{\omega}{m}.$$

Here, we omit the index of $\mu, \nu, \lambda, \sigma$ in $Q^{\mu\nu\lambda\sigma}, R^{\mu\nu\lambda\sigma}, S^{\mu\nu\lambda\sigma}, T^{\mu\nu\lambda\sigma}$. Therefore, we find

$$\begin{aligned} \bar{M}_a^{\mu\nu\lambda\sigma} = & -\frac{(ie)^4}{(2\pi)^4} i\pi^2 \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \left\{ \frac{-8m^2 Q - \omega^2 R}{-m^2(1 - \xi^2 U)} + \frac{\omega^4 S + m^2 \omega^2 R + m^4 T}{(-m^2(1 - \xi^2 U))^2} \right. \\ & \left. + 8Q \left(\ln \left| \frac{\Lambda^2}{m^2} \right| - \ln |1 - \xi^2 U| - \frac{11}{6} \right) \right\} \end{aligned}$$

$$\begin{aligned} \bar{M}_a^{\mu\nu\lambda\sigma} = & -\frac{(ie)^4}{(2\pi)^4} i\pi^2 \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \left\{ \frac{8Q + \xi^2 R}{1 - \xi^2 U} + \frac{\xi^4 S + \xi^2 R + T}{(1 - \xi^2 U)^2} \right. \\ & \left. + 8Q \left(\ln \left| \frac{\Lambda^2}{m^2} \right| - \ln |1 - \xi^2 U| - \frac{11}{6} \right) \right\} \end{aligned}$$

Expanding in terms of $\xi = \frac{\omega}{m}$, we find

$$\begin{aligned} \bar{M}_a^{\mu\nu\lambda\sigma} = & -\frac{(ie)^4}{(2\pi)^4} i\pi^2 \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \left\{ 8Q + (R + 8QU)\xi^2 + (RU + 8QU^2)\xi^4 + \dots \right. \\ & \left. + T + (R + 2TU)\xi^2 + (S + 2RU + 3TU^2)\xi^4 + \dots \right. \end{aligned}$$

$$\left. + 8Q \left(\ln \left| \frac{\Lambda^2}{m^2} \right| + U\xi^2 + \frac{1}{2}\xi^4 U^2 + \dots - \frac{11}{6} \right) \right\}$$

$$\bar{M}_a^{\mu\nu\lambda\sigma} = -\frac{(ie)^4}{(2\pi)^4} i\pi^2 \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \left\{ 8Q + 8Q \left(\ln \left| \frac{\Lambda^2}{m^2} \right| - \frac{11}{6} \right) + T + \dots \right\}$$

The coefficient of Q at the lowest order can be written as $8 + 8 \left(\ln \left| \frac{\Lambda^2}{m^2} \right| - \frac{11}{6} \right)$, which does not depend on the shape of M_a, M_b, M_c . Therefore, we find

$$\bar{M}_a^{\mu\nu\lambda\sigma} \cong -\frac{(ie)^4}{(2\pi)^4} i\pi^2 \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 T = -\frac{(ie)^4}{(2\pi)^4} i\pi^2 \frac{1}{6} T$$

and we can write it explicitly as

$$\bar{M}_a^{\mu\nu\lambda\sigma} = -\frac{(ie)^4}{(2\pi)^4} i\pi^2 \frac{1}{6} T^{\mu\nu\lambda\sigma}.$$

VI. TOTAL AMPLITUDE

In this way, we can obtain the shape of $\bar{M}_a^{\mu\nu\lambda\sigma}$, $\bar{M}_b^{\mu\nu\lambda\sigma}$ and $\bar{M}_c^{\mu\nu\lambda\sigma}$ at $\frac{\omega}{m} \ll 1$. Thus, we can write for M_a , M_b , M_c as

$$M_a = -\frac{(ie)^4}{(2\pi)^4} i\pi^2 \frac{1}{6} \left\{ 4(g^{\mu\sigma} g^{\lambda\nu} + g^{\mu\nu} g^{\sigma\lambda} - g^{\mu\lambda} g^{\sigma\nu}) \right\} \epsilon_\mu^r \epsilon_\nu^s \epsilon_\lambda^{s'} \epsilon_\sigma^{r'} \quad (5.1.a)$$

$$M_b = -\frac{(ie)^4}{(2\pi)^4} i\pi^2 \frac{1}{6} \left\{ 4(g^{\mu\lambda} g^{\sigma\nu} + g^{\mu\nu} g^{\sigma\lambda} - g^{\mu\sigma} g^{\lambda\nu}) \right\} \epsilon_\mu^r \epsilon_\nu^s \epsilon_\lambda^{s'} \epsilon_\sigma^{r'} \quad (5.1.b)$$

$$M_c = -\frac{(ie)^4}{(2\pi)^4} i\pi^2 \frac{1}{6} \left\{ 4(g^{\mu\sigma} g^{\lambda\nu} + g^{\sigma\nu} g^{\mu\lambda} - g^{\sigma\lambda} g^{\mu\nu}) \right\} \epsilon_\mu^r \epsilon_\nu^s \epsilon_\lambda^{s'} \epsilon_\sigma^{r'}. \quad (5.1.c)$$

Thus, the total amplitude can be calculated as

$$\begin{aligned} \frac{1}{2}M &= M_a + M_b + M_c = -\frac{(ie)^4}{(2\pi)^4} i\pi^2 \frac{1}{6} 4 \left\{ g^{\mu\sigma} g^{\lambda\nu} + g^{\mu\nu} g^{\sigma\lambda} - g^{\mu\lambda} g^{\sigma\nu} \right. \\ &\quad \left. g^{\mu\lambda} g^{\sigma\nu} + g^{\mu\nu} g^{\sigma\lambda} - g^{\mu\sigma} g^{\lambda\nu} \right. \\ &\quad \left. g^{\mu\sigma} g^{\lambda\nu} + g^{\sigma\nu} g^{\mu\lambda} - g^{\sigma\lambda} g^{\mu\nu} \right\} \epsilon_\mu^r \epsilon_\nu^s \epsilon_\lambda^{s'} \epsilon_\sigma^{r'} \\ &= -\frac{(ie)^4}{(2\pi)^4} i\pi^2 \frac{1}{6} 4 \left(g^{\mu\sigma} g^{\lambda\nu} + g^{\mu\nu} g^{\sigma\lambda} + g^{\mu\lambda} g^{\sigma\nu} \right) \epsilon_\mu^r \epsilon_\nu^s \epsilon_\lambda^{s'} \epsilon_\sigma^{r'} \end{aligned}$$

Therefore, we can write M as

$$M = -i \frac{4}{3} \alpha^2 \left(g^{\mu\sigma} g^{\lambda\nu} + g^{\mu\nu} g^{\sigma\lambda} + g^{\mu\lambda} g^{\sigma\nu} \right) \epsilon_\mu^r \epsilon_\nu^s \epsilon_\lambda^{s'} \epsilon_\sigma^{r'} \quad (5.2)$$

where $\alpha = \frac{e^2}{4\pi}$.

VII. PHOTON-PHOTON SCATTERING

Now, the photon-photon scattering cross section can be written as [4]

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{1}{(2\omega)^2} |M|^2 \quad (6.1)$$

where we sum up the final states of the photon polarization state and make average of the initial polarization states.

$$|M|^2 = \frac{1}{4} \sum_{rr'ss'} |M^{rr'ss'}|^2.$$

Among the above, the nonvanishing term is

$$\begin{aligned} &= \frac{1}{4} \left\{ |M^{1111}|^2 + |M^{2222}|^2 + |M^{1122}|^2 + |M^{2211}|^2 \right. \\ &\quad \left. + |M^{1221}|^2 + |M^{2112}|^2 + |M^{1212}|^2 + |M^{2121}|^2 \right\} \end{aligned}$$

Since we know that

$$M^{1122} = M^{2211} \quad M^{1221} = M^{2112} \quad M^{1212} = M^{2121}$$

we obtain

$$\frac{1}{4} \sum_{rr'ss'} |M^{rr'ss'}|^2 = \frac{1}{4} \left\{ |M^{1111}|^2 + |M^{2222}|^2 + 2|M^{1122}|^2 + 2|M^{1221}|^2 + 2|M^{1212}|^2 \right\}.$$

Further, we find

$$M^{1111} = -i \frac{4}{3} \alpha^2 \cdot 3$$

$$M^{2222} = -i \frac{4}{3} \alpha^2 (1 + 2 \cos^2 \theta)$$

$$M^{1122} = -i \frac{4}{3} \alpha^2 \cos \theta$$

$$M^{1221} = i \frac{4}{3} \alpha^2 \cos \theta$$

$$M^{1212} = i \frac{4}{3} \alpha^2 \cdot 1$$

and thusA

$$\frac{1}{4} \sum |M|^2 = \frac{16}{9} \alpha^2 \{3 + 2 \cos^2 \theta + \cos^4 \theta\}.$$

Therefore, for $\frac{\omega}{m} \ll 1$, we find

$$\frac{d\sigma}{d\Omega} = \frac{1}{(6\pi)^2} \frac{\alpha^4}{(2\omega)^2} (3 + 2 \cos^2 \theta + \cos^4 \theta) \quad (6.2)$$

where $\alpha \equiv \frac{e^2}{4\pi}$. This is the scattering cross section in $\frac{\omega}{m} \ll 1$. This result is very different from the one obtained by Euler-Heisenberg [3],[4]. The calculated result of Euler-Heisenberg can be written as

$$\frac{d\sigma}{d\Omega} = \frac{139\alpha^4}{(180\pi)^2 m^2} \left(\frac{\omega}{m}\right)^6 (3 + \cos^2 \theta)^2. \quad (6.3)$$

In particular, the energy dependence of the cross section is completely different from each other. At the low energy limit, the new cross section becomes larger while the Euler-Heisenberg cross section becomes smaller. At $\omega \simeq 1$ eV, the Euler-Heisenberg cross section with 90 degree becomes

$$\frac{d\sigma}{d\Omega} = \frac{139\alpha^4}{(60\pi)^2 m^2} \left(\frac{\omega}{m}\right)^6 \simeq 9.3 \times 10^{-67} \text{ cm}^2 \simeq 9.3 \times 10^{-43} b$$

which is extremely small. This suggests that the Euler-Heisenberg cross section is practically impossible to measure since it is too small. On the other hand, the present calculation predicts the cross section at $\omega \simeq 1$ eV

$$\frac{d\sigma}{d\Omega} = \frac{3\alpha^4}{(12\pi)^2 \omega^2} \simeq 2.3 \times 10^{-21} \text{ cm}^2 \simeq 2.3 \times 10^3 b.$$

This is rather large, and as far as the magnitude of the cross section is concerned, it should be indeed measurable. These problems are discussed in detail in [9].

VIII. WHY IS EULER-HEISENBERG LAGRANGIAN INCORRECT ?

Here, we briefly describe the physical reason why the Euler-Heisenberg is incorrect.

A. Euler-Heisenberg Lagrangian density

The Euler-Heisenberg Lagrangian density is given as

$$\mathcal{L} = \frac{2\alpha^2}{45m^4} \left[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 7(\mathbf{E} \cdot \mathbf{B})^2 \right]. \quad (7.1.a)$$

They first calculate the vacuum polarization effects when they apply the electromagnetic fields to the Dirac vacuum states which are composed out of negative energy fermions. In this case, they obtained the effects of the vacuum polarization, and constructed the effective Lagrangian density which can simulate the results. The basic problem is that they treated the vacuum state as if it were dielectric matter.

B. Karplus and Neuman's calculation

Now, we discuss the result calculated by Karplus and Neuman who started their calculation from the QED Lagrangian density. Here, we want to clarify why Karplus and Neuman obtained the same results as that of Euler-Heisenberg, in spite of the fact that Karplus and Neuman employed the modern field theory terminology. First, we rewrite the Euler-Heisenberg Lagrangian density as

$$\mathcal{L} = -\frac{\alpha^2}{180m^4} \left\{ 5(F_{\mu\nu}(x)F^{\mu\nu}(x))^2 - 14F_{\mu\nu}(x)F^{\nu\lambda}(x)F_{\lambda\sigma}(x)F^{\sigma\mu}(x) \right\} \quad (7.1.b)$$

where

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x).$$

Karplus and Neuman calculated first the QED evaluation based on the QED Lagrangian density. However, they always imposed the conditions that they should be able to reproduce the results of the Euler-Heisenberg calculation. This must be a mystery why they believed that they should get the same result as the one by Euler-Heisenberg. However, this may well be connected to the additional conditions which are often called "gauge condition" even though there is no physical reason for this gauge condition [10].

IX. CONCLUSION

We have presented the new QED calculation of photon-photon scattering at low energy. The calculation is straightforward since there is no logarithmic divergence in the evaluation of the photon-photon scattering Feynman diagrams. The result is very different from the Euler-Heisenberg calculation, and therefore the photon-photon cross section should be measured by experiments in order to clarify which of the cross section should be preferred by nature.

X. ACKNOWLEDGEMENTS

The author is grateful to Prof. T. Fujita for his helpful discussions and comments.

-
- [1] H. Euler and B. Kockel, *Naturwissensch.* **23**, 246 (1935)
 - [2] H. Euler, *Ann. der. Physik.* **26**, 398 (1936)
 - [3] W. Heisenberg and H. Euler, *Z.Phys.* **98**, 714 (1936)
 - [4] V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii, "Relativistic Quantum Theory", (Pergamon Press, 1974)

- [5] R. Karplus and M. Neuman, Phys. Rev. **80**, 380 (1950)
- [6] R. Karplus and M. Neuman, Phys. Rev. **83**, 776 (1951)
- [7] K. Nishijima, "Fields and Particles", (W.A. Benjamin, INC, 1969)
- [8] F. Mandl and G. Shaw, "Quantum Field Theory", (John Wiley & Sons, 1993)
- [9] T. Fujita and N. Kanda, A Proposal to Measure Photon-Photon Scattering (to be published)
- [10] T. Fujita and N. Kanda, "Tomonaga's Conjecture on Photon Self-Energy", physics.gen-ph/1102.2974